Contributions of Lattice Theory to the Study of Computational Topology

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in a joint work with

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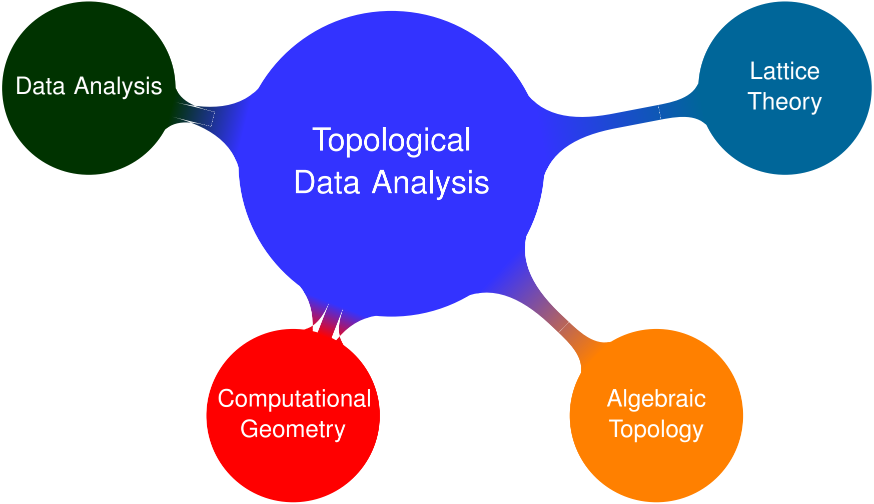
AAA88 Conference,

Warsaw, June 20, 2014

www.joaopitacosta.com/aaa88

IJS   Artificial Intelligence Laboratory   SEVENTH FRAMEWORK PROGRAMME

Motivations



Topological Data Analysis

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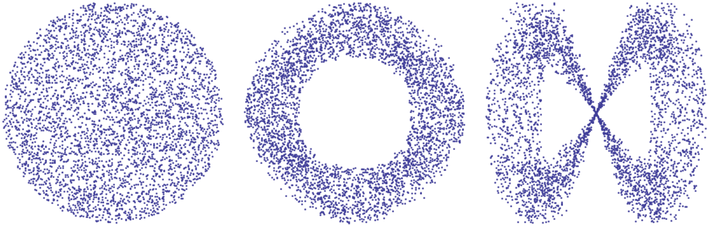

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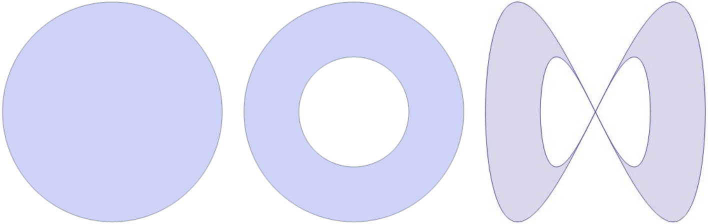

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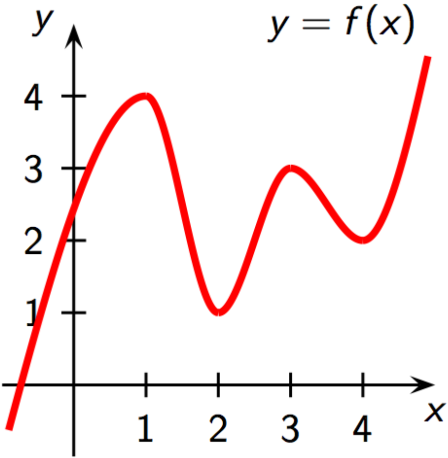

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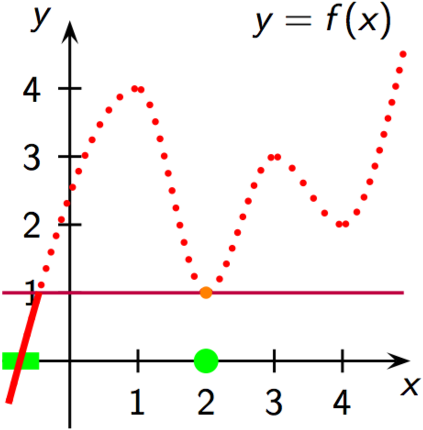
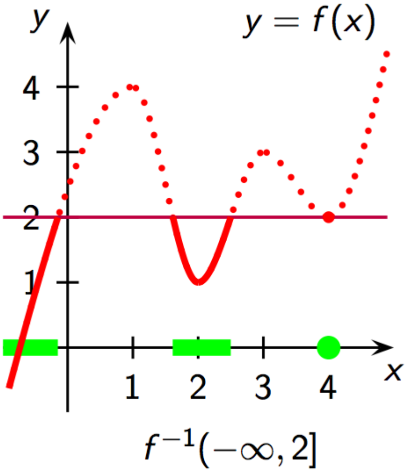


$f^{-1}(-\infty, 1]$

Topological Data Analysis



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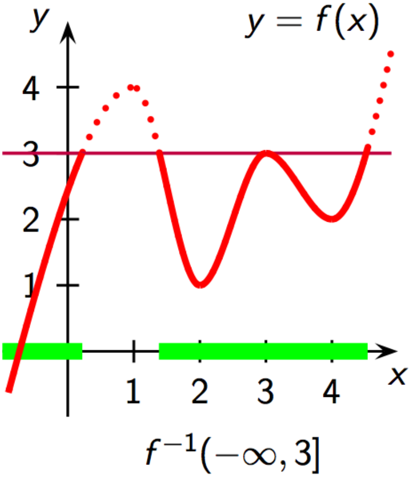

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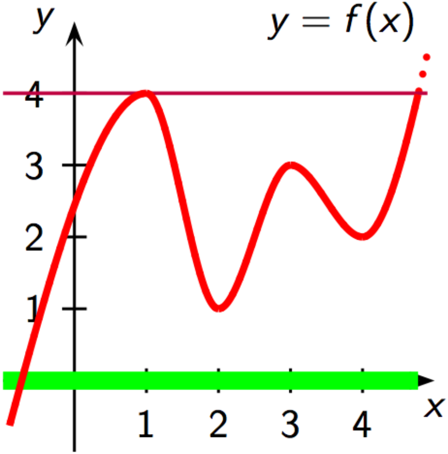
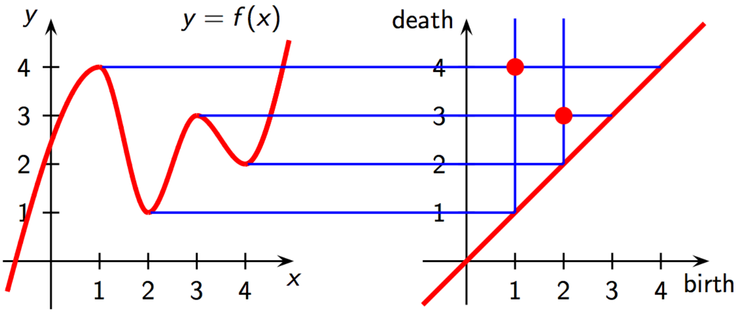


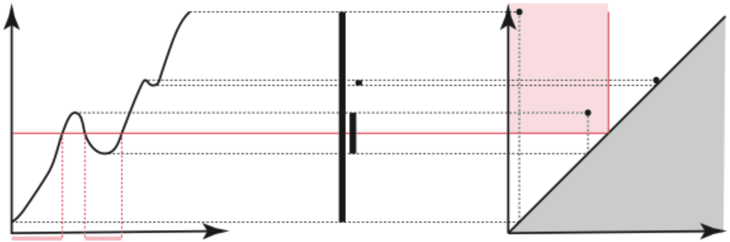
$f^{-1}(-\infty, 4]$

Topological Data Analysis



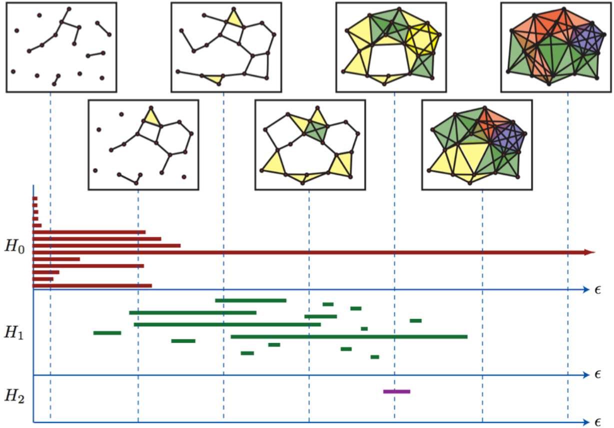
Persistent Homology

Persistence of H_0 of sublevel-sets of a real function.



Mikael Vejdemo-Johansson, Sketches of a platypus: persistence homology and its foundations. arXiv:1212.5398v1 (2013)

Persistent Homology



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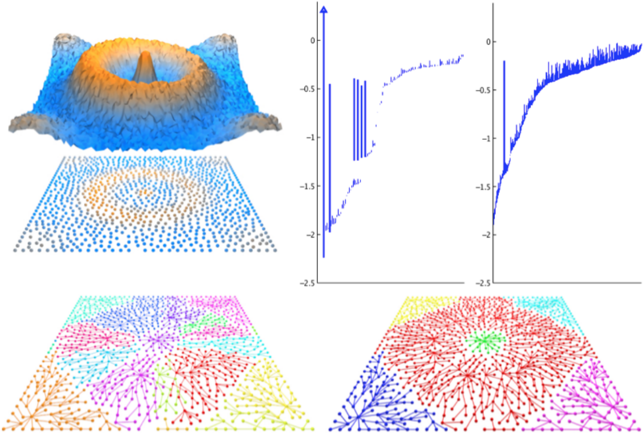
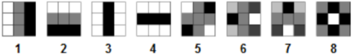
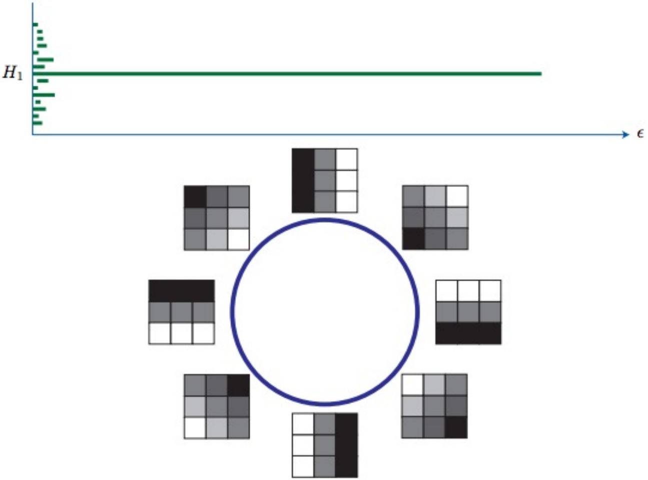


Persistent Homology

Application: Image Analysis

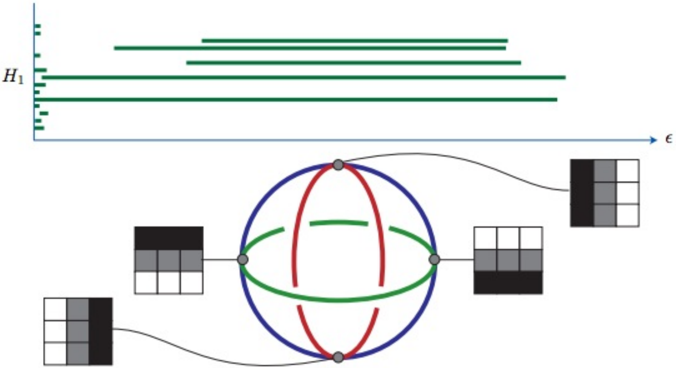


Persistent Homology



R. Ghrist, Barcodes: the persistent topology of data. Bulletin of the American Math. Soc. 45.1 (2008): 61-75.

Persistent Homology



R. Ghrist, Barcodes: the persistent topology of data. Bulletin of the American Math. Soc. 45.1 (2008): 61-75.

Persistent Homology

Application: Tumor Detection

Heyting Algebras

A **Boolean algebra** $(L; \wedge, \vee, \neg, 0, 1)$ is a distributive lattice $(L; \wedge, \vee)$ with bounds 0 and 1 such that all elements $x \in L$ have complement y (noted $\neg x$) satisfying $x \wedge y = 0$ and $x \vee y = 1$.

A **Heyting algebra** $(L; \wedge, \vee, \Rightarrow)$ is a distributive lattice $(L; \wedge, \vee)$ such that for each pair $a, b \in L$ there is a greatest element $x \in L$ (noted $a \Rightarrow b$) such that $a \wedge x \leq b$. The **pseudo complement** of $x \in L$ is $x \Rightarrow 0$ (often also noted by $\neg x$).

Example

Every Boolean algebra is a Heyting algebra with $a \Rightarrow b = \neg a \vee b$ and $a \Rightarrow 0 = \neg a$. The open sets of a topological space X constitute a complete Heyting algebra with $A \Rightarrow B = int((X - A) \cup B)$.

Heyting Algebras

The collection of all open subsets of a topological space X forms a complete Heyting algebra.

Heyting algebra H	**Topological space** X
$U \wedge V$	$U \cap V$
$U \vee V$	$U \cup V$
0	$\emptyset$
1	X
$U \Rightarrow V$	$int((X - U) \cup V$
$\neg U$	$int(X - U)$

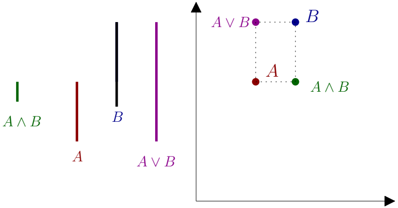
Algebra of Lifetimes

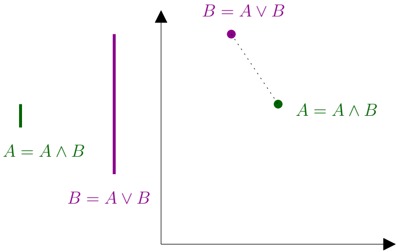
Mikael Vejdemo-Johansson, Sketches of a platypus: persistence homology and its foundations. arXiv:1212.5398v1 (2013)

Algebra of Lifetimes

Definition. Consider the complete lattice $(\mathrm{R}; \wedge, \vee)$. Let A and B be intervals $A = \mathcal{B}(a_1, a_2)$ and $B = \mathcal{B}(b_1, b_2)$ represented in a persistence diagram by the points $A(a_1, a_2)$ and $B(b_1, b_2)$ in $\mathcal{H}$. Define:

$$A \wedge B = (a_1 \vee b_1, a_2 \wedge b_2) \text{ and } A \vee B = (a_1 \wedge b_1, a_2 \vee b_2)$$





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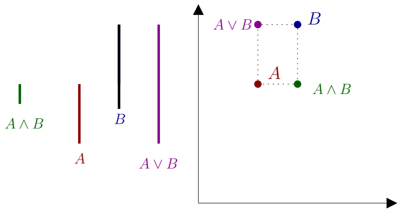
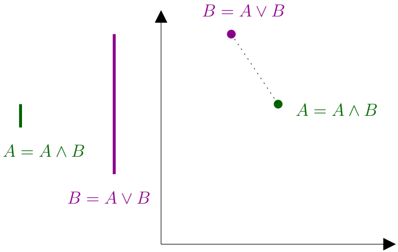


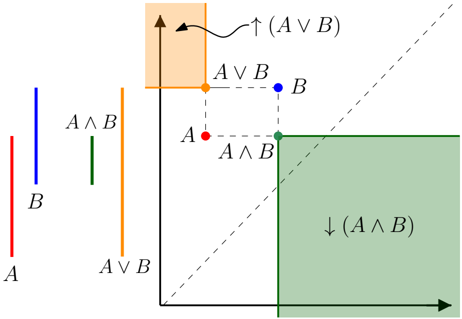


$\mathcal{H}$ is a Heyting algebra.

Algebra of Lifetimes

Ordering bars in $\mathcal{H}$

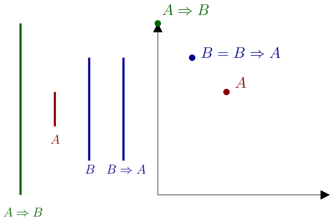
$$A \leq B \text{ iff } A \wedge B = A \text{ iff } b_1 \leq a_1 \text{ and } a_2 \leq b_2 \text{ iff } \mathcal{B}(A) \subseteq \mathcal{B}(B).$$

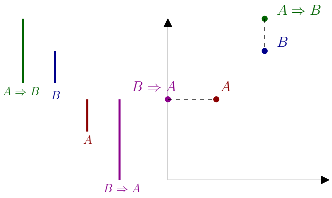


Algebra of Lifetimes

$$\text{If } A \leq B \text{, then } A \Rightarrow B = \begin{cases} 1 = (0, \varepsilon_2) & \text{, if } b_1 \leq a_1 \text{ and } a_2 \leq b_2 \\ B = (b_1, b_2) & \text{, if } a_1 \leq b_1 \text{ and } b_2 \leq a_2 \end{cases}.$$

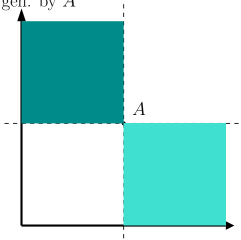
$$\text{Otherwise, } A \Rightarrow B = \begin{cases} (b_1, \varepsilon_2) & \text{, if } a_1 \leq b_1 \text{ and } a_2 \leq b_2 \\ (0, b_2) & \text{, if } b_1 \leq a_1 \text{ and } b_2 \leq a_2 \end{cases}.$$





Ideals and Filters

Assuming that $\mathcal{H}$ is bounded by $(0,0)$ and $(\varepsilon_1, \varepsilon_2)$:



Filter gen. by a bar A: $\uparrow A = [0, a_1] \times [a_2, \varepsilon_2]$.
Ideal gen. by a bar A: $\downarrow A = [a_1, \varepsilon_1] \times [0, a_2]$.

Algebra of Lifetimes

The poset $\mathcal{H}$ together with the operations

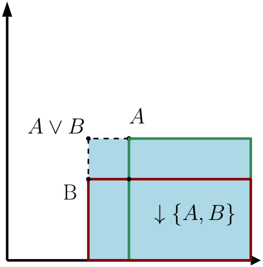
$$\bigwedge_i A_i = (\bigvee\{\, a_{1i} \,\}, \bigwedge\{\, a_{2i} \,\}) \text{ and } \bigvee_i A_i = (\bigwedge\{\, a_{1i} \,\}, \bigvee\{\, a_{2i} \,\}).$$

is a complete Heyting algebra. In particular, $\mathcal{H}$ is completely distributive, i.e., the following identity holds

$$X \wedge \bigvee_{i \in I} Y_i = \bigvee_{i \in I} (X \wedge Y_i).$$

Algebra of Lifetimes

Ideal gen. by bars A and B: $[min\{\, a_1, b_1 \,\}, \varepsilon_1] \times [0, \max\{\, a_2, b_2 \,\}]$.
Filter gen. by bars A and B: $[0, \max a_1, b_1] \times [\min\{\, a_2, b_2 \,\}, \varepsilon_2]$.



Ideal gen. by a family $\{\, A_i \,\}_{i \in I}$:
$\downarrow \{\, A_i \,\}_{i \in I} = \downarrow \bigvee_{i \in I} A_i = [\min\{\, a_i \,\}, \varepsilon_1] \times [0, \max\{\, a_i \,\}]$
Filter gen. by a family $\{\, A_i \,\}_{i \in I}$:
$\uparrow \{\, A_i \,\}_{i \in I} = \uparrow \bigwedge_{i \in I} A_i = [0, \max\{\, a_i \,\}] \times [\min\{\, a_i \,\}, \varepsilon_2]$.

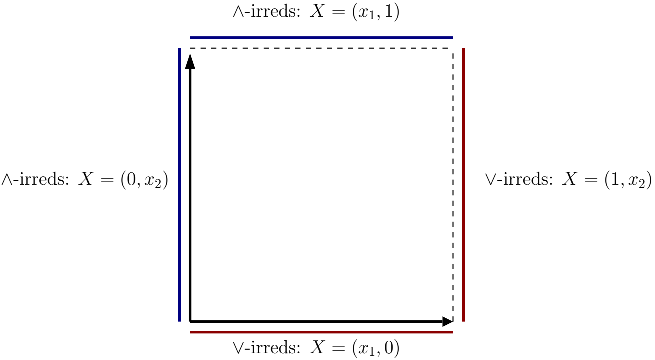
Algebra of Lifetimes

x is **join-irreducible** if $x = y \vee z$ implies $x = y$ OR $x = z$
x is **meet-irreducible** if $x = y \wedge z$ implies $x = y$ OR $x = z$

Join-irreducibles of $\mathcal{H}$: bars with coordinates $(x_1, 0)$ or (ε_1, x_2).
Meet-irreducibles of $\mathcal{H}$: bars with coordinates $(0, x_2)$ or (x_1, ε_2).

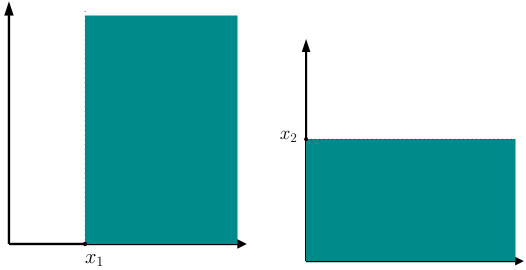
Algebra of Lifetimes

Join-irreducibles of $\mathcal{H}$: bars with coordinates $(x_1, 0)$ or (ε_1, x_2).
Meet-irreducibles of $\mathcal{H}$: bars with coordinates $(0, x_2)$ or (x_1, ε_2).



Algebra of Lifetimes

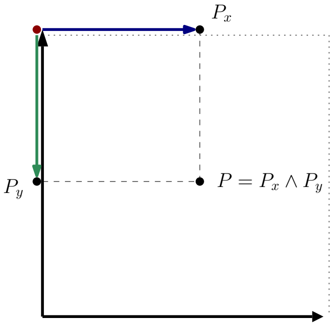
An ideal I of L is a **prime ideal** if for all $x, y \in L$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$.
Join-irreducibles of $\mathcal{H}$: bars with coordinates $(x_1, 0)$ or $(1, x_2)$.
$\downarrow(x_1, \varepsilon_2) = [x_1, \varepsilon_1] \times [0, \varepsilon_2]$ and $\downarrow(0, x_2) = [0, \varepsilon_1] \times [x_2, \varepsilon_2]$ are prime ideals of $\mathcal{H}$.



The Dual Space

Spac := Spatial Locales and homomorphisms $\cong$ **Sob** := Sober Spaces and homeomorphisms

An open of the dual space as the sum of two filters intersecting only in $\top$ and the point in the lattice it corresponds to.



The Dual Space

$$
\begin{tikzcd}
\textbf{Ring R} \arrow[rr] & & \textbf{Spec(R)} \arrow[ll, bend right, dashed, "\text{scheme}"'] \arrow[d, leftrightarrow, "?"] \\
\textbf{Locale H} \arrow[rr] & & \textbf{Spec(H)} \arrow[ll, bend left, dashed, "\text{skew distributive lats}"]
\end{tikzcd}
$$

The Dual Space

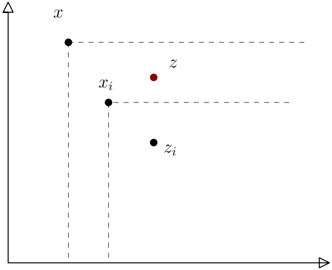
$$\begin{tikzcd} PS_L \arrow[r, leftrightarrow, "\text{P1972}"] \arrow[d] & DL_0 \arrow[d] \\ Sh(PS_L) \arrow[r, leftrightarrow, "\text{CBKMG 2012}"'] & \text{SDL} \end{tikzcd}$$

Topos of Sheaves over $\mathcal{H}$

Consider the functor $\phi : \mathcal{H} \to Set$ defined by the sections

$$\phi(x) = \{\downarrow y \mid y \leq x\},$$

for all $x \in \mathcal{H}$, and the restriction map $\chi^x_y : \phi(x) \to \phi(y)$ defined by $\chi^x_y(\downarrow \sigma) = (\downarrow z) \cap (\downarrow y)$ for all $x, y, z \in \mathcal{H}$ such that $y \leq x$.



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$$\begin{aligned}
\downarrow z \cap \downarrow x_i &= (\bigvee_{j \in J} \downarrow z_j) \cap \downarrow x_i \\
&= \bigvee_{j \in J} (\downarrow z_j \cap \downarrow x_i) \\
&= \bigvee_{j \in J} \downarrow z_i \cap \downarrow x_j \\
&= \downarrow z_i \cap (\bigvee_{j \in J} \downarrow x_j) \\
&= \downarrow z_i \cap \downarrow x = \downarrow z_i
\end{aligned}$$

Topos of Sheaves over $\mathcal{H}$

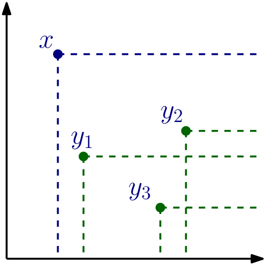
The category of presheaves $Set^{\mathcal{H}^{op}}$ has exponentials: the exponential object z^y is the implication $y \Rightarrow z$.
The topos of sheaves on $\mathcal{H}$ is given by the subject classifier including only the closed sieves:

$$\Omega(x) = \{\downarrow y \mid y \leq x\}.$$

Topos of Sheaves over $\mathcal{H}$

$$
\begin{tikzcd}
U \arrow[r] \arrow[d, "\exists ! \phi"'] & 1 \arrow[d, "true"] \\
E \arrow[r, "\chi_E"'] & \Omega
\end{tikzcd}
$$

$$\Omega(x) = \{ \downarrow y \mid y \leq x \}.$$



Topos of Sheaves over $\mathcal{H}$

$$\begin{tikzcd} \Delta \arrow[r] \arrow[dr] & Set \arrow[d] \\ & Set^{H^{op}} \end{tikzcd}$$

Open Problems

- Computation of semisimplicial homology on the topos over $\mathcal{H}$;
- Study of the dual space and respective spectral space;
- Interpretation of the arrow operation in the framework;
- Integration of classical persistence results under this perspective;
- Implementation of new algorithms.

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